Holographic Quantum States

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We show how continuous matrix product states of quantum fields can be described in terms of the dissipative nonequilibrium dynamics of a lower-dimensional auxiliary boundary field by demonstrating that the spatial correlation functions of the bulk field correspond to the temporal statistics of the boundary field. This equivalence (1) illustrates an intimate connection between the theory of continuous quantum measurement and quantum field theory, (2) gives an explicit construction of the boundary field allowing the extension of real-space renormalization group methods to arbitrary dimensional quantum field theories without the introduction of a lattice parameter, and (3) yields a novel interpretation of recent cavity QED experiments in terms of quantum field theory, and hence paves the way toward observing genuine quantum phase transitions in such zero-dimensional driven quantum systems.

In recent years we have witnessed tremendous success in the calculation of physical properties of quantum many-body systems from their wave functions. These developments have spurred the study of the quantum entanglement in these models, resulting in the realization that natural states of quantum lattice systems are only slightly entangled, and hence obey an entropy area law [1–4]. As a consequence, we now understand that physical lattice states are well-captured by matrix product states (MPS) [5].

These developments have also allowed the interpretation of the renormalization methods of Wilson [6] and White [7,8] as applications of the variational principle to matrix product states, and have led to natural generalizations of these renormalization group schemes to higher dimensions [9–12]. In these schemes the quantum correlation properties of natural lattice states are encoded in the variational parameters of an auxiliary zero-dimensional system.

A natural next step is then to develop a similar approach for quantum fields, and this is exactly the subject of this paper. However, capturing the manifold of low-energy wave functionals is much more challenging due to the continuous infinity of degrees of freedom. The most natural way to proceed is to discretize the continuous degrees of freedom via a lattice cutoff and truncate the local Hilbert spaces at each site [13–15], like in lattice gauge theory which, so far, provides essentially the only systematic way to understand nonperturbative effects. Recently, however, it was established that there is no need to impose a lattice cutoff because continuum limits of matrix product states can be directly defined, and these states, termed continuous matrix product states (cMPS), can represent the physics of nonrelativistic field theories accurately [16,17].

The first result of this work is a procedure to generate cMPS, providing a natural physical interpretation of this variational class. Our procedure is based on the paradigm of continuous measurement [18] and is a generalization of the sequential preparation scheme for MPS proposed in [19] to the continuous setting. This perspective allows one to easily design more flexible variational classes of states, including quantum states for bosonic systems at arbitrary filling. Our approach also suggests a natural and powerful generalization to arbitrary dimensions, yielding our second main result: a variational class for $(2 + 1)$-dimensional quantum field theories. The boundary field here provides a local parametrization of the bulk field realizing one of the major prerequisites identified by Feynman for the successful application of the variational principle to quantum field theory [20].

Surprisingly, the bulk and boundary fields have a direct interpretation in the context of cavity electrodynamics [19,21–23]: the role of the auxiliary system is played by the cavity modes and the quantum field describes the photons leaking from the cavity. An atom with a fairly low number of internal addressable levels (e.g., $D = 6$), would already allow the reproduction of all static correlations functions in, e.g., the Lieb-Liniger model [24]. This is achieved by observing the temporal counting statistics of the photons leaking from the cavity. The present paper also sheds new light on the recently discovered phase transitions of the quantum trajectories obtained in dissipative systems [25]: such dynamical phase transitions are in correspondence with static quantum phase transitions of a quantum field theory in one dimension higher.

We begin by modeling the measurement of some physical observable M on a D-level quantum system, which we initially call the "system." Our model, known as von Neumann's prescription [26], is defined as follows. We attach a quantum system with a continuous degree of freedom, called the meter, in a fiducial state vector $|0\rangle$ and couple it with the system for some time t according to the interaction $H_I = M \otimes p$. Supposing the system is initially in $|\phi\rangle$, then after the interaction the state is

$e^{-itH_I}|\phi\rangle|0\rangle = \sum_{j=1}^{D} \phi_j|m_j\rangle|x = m_j t\rangle$, where $|m_j\rangle$ are the eigenvectors of M with corresponding eigenvalues m_j and $|\phi\rangle = \sum_{j=1}^{D} \phi_j|m_j\rangle$. We never actually perform a projective measurement of the meter; i.e., the measurement record is discarded.

The core of our proposal is to turn von Neumann's measurement prescription on its head and regard the meter (attached as an ancillary system) as the fundamental system $\mathcal{A}$ and the system as an auxiliary ancilla $\mathcal{B}$. In this way we can think of it as a state generation device: we obtain a variety of physical quantum states of the meter $\mathcal{A}$ alone by exploiting the measurement prescription and then tracing out, or perhaps measuring, the system $\mathcal{B}$. We thus have a way to generate states of a quantum system with a continuous degree of freedom. The challenge remains, however, to exploit this procedure to obtain quantum states of a continuous infinity of such continuous degrees of freedom.

The way we do this here is to model a continuous measurement process. Our model, which generalizes Ref. [18], is defined by a family of $D \times D$ complex matrices $R(x)$, $x \in [0, L]$, which we instantaneously and infinitely weakly measure on $\mathcal{B}$ at time $t = x$, which is additionally evolving according to some free Hamiltonian $K(x)$. We do this by introducing a collection $\mathcal{A}$ of n meters, labeled by $r = 1, 2, \ldots, n$. The total Hamiltonian is given by

$$H(t) = K(t) \otimes \mathbb{1}_{\mathcal{A}} + H_I(t), \tag{1}$$

where $H_I(t) = \sqrt{\epsilon} \sum_{r=1}^{n} \delta(t - r\epsilon)[iR(r\epsilon) \otimes a_{r\epsilon}^{\dagger} + \text{H.c.}]$. We are interested in the limit where $n \to \infty$ and $\epsilon \to 0$ with $n\epsilon = L$ fixed. (For finite n, this approach includes the scheme of Ref. [13].) The choice of the coefficient $\sqrt{\epsilon}$ in the definition of H_I can be understood on physical grounds: any other scaling would lead to trivial dynamics, thanks to the quantum Zeno effect, or to the situation where the meters and system do not interact [18].

We now supply a physical interpretation of the cMPS of [16] based on the sequential preparation prescription of [19] and a continuous measurement scenario. Integrating the Schrödinger equation for Eq. (1) and making the standard definition of the discretized quantum field operator as $\Psi(r\epsilon) = a_{r\epsilon}/\sqrt{\epsilon}$ we find, in the limit $\epsilon \to 0$, that

$$U(L) = \mathcal{T} e^{-i \int_0^L [K(s) \otimes \mathbb{1}_{\mathcal{A}} + iR(s) \otimes \Psi^{\dagger}(s) - iR^{\dagger}(s) \otimes \Psi(s)]ds}. \tag{2}$$

In our continuum limit the collection $\mathcal{A}$ of the meters may be regarded as a geometrically one-dimensional bosonic quantum field. The auxiliary system $\mathcal{B}$ may be regarded as a geometrically zero-dimensional bosonic quantum field.

We now learn that the evolution Eq. (2) prepares cMPS. Indeed, if we initialize the meters $\mathcal{A}$ in the vacuum $|\Omega\rangle_{\mathcal{A}}$ we can, for each r, exploit the Baker-Hausdorff formula to see that $e^{\sqrt{\epsilon}[R(r\epsilon) \otimes a_{r\epsilon}^{\dagger} - R^{\dagger}(r\epsilon) \otimes a_{r\epsilon}]}(\mathbb{1}_{\mathcal{B}} \otimes |0\rangle_{\mathcal{A}}) = e^{-(\epsilon/2)R^{\dagger}(r\epsilon)R(r\epsilon)} e^{\sqrt{\epsilon}R(r\epsilon) \otimes a_{r\epsilon}^{\dagger}}(\mathbb{1}_{\mathcal{B}} \otimes |0\rangle_{\mathcal{A}})$ is valid to $O(\epsilon)$ to rewrite the limit Eq. (2) for $U(L)(\mathbb{1}_{\mathcal{B}} \otimes |\Omega\rangle_{\mathcal{A}})$ as

$$\mathcal{T} e^{\int_0^L [Q(s) \otimes \mathbb{1}_{\mathcal{A}} + R(s) \otimes \Psi^{\dagger}(s)]ds}(\mathbb{1}_{\mathcal{B}} \otimes |\Omega\rangle_{\mathcal{A}}),$$

where $Q(s) = -iK(s) - \frac{1}{2}R^{\dagger}(s)R(s)$. This is equivalent to the definition of a cMPS [16].

Thus, a cMPS is a state of the quantum field $\mathcal{A}$ preparable via continuous measurement; i.e., we initialize the quantum field $\mathcal{A}$ in some (known) prespecified quantum state $\omega_{\mathcal{A}} = |\Omega\rangle_{\mathcal{A}}\langle\Omega|$ and adjoin an auxiliary zero-dimensional quantum field $\mathcal{B}$ initialized in some (possibly mixed) fiducial state ρ. We then interact $\mathcal{A}$ and $\mathcal{B}$ according to the manifestly unitary continuous measurement dynamics $U(L)$. We then discard the auxiliary system.

We pause here to emphasize an important point: when the initial state $\omega_{\mathcal{A}}$ of the quantum field $\mathcal{A}$ is taken to be the vacuum then the states generated by our procedure are states on Fock space built from the single-particle space $L^2([0, L])$. (Physically this can be explained as follows: the interaction H_I between $\mathcal{A}$ and $\mathcal{B}$ can only transport ϵ bosons to $\mathcal{A}$ in a time ϵ. Thus the interaction can never achieve superpositions of terms with more than a constant number of bosons per unit length.) However, our construction is much more flexible as all we require is that the initial field state arises from the continuum limit of a product state $\omega_r \otimes \cdots \otimes \omega_1$ of the meters $r = 1, \ldots, n$. Thus, crucially, we can allow for initial field states with a high density of bosons and superpositions of bosons; in this case the states generated by the sequential preparation procedure will no longer be elements of Fock space. This will be important in a variety of contexts, particularly those pertaining to dense systems with nonlinear interactions, where a cMPS defined using the empty vacuum will be insufficient.

We now investigate the dynamics of the auxiliary system throughout the continuous measurement process. Again, we first consider the discrete setting and take the continuum limit. We set $\rho(0) = \rho_{\mathcal{B}}$ and $\rho(r\epsilon) = \text{tr}_{\mathcal{A}}[U(r\epsilon) \times (\rho(0) \otimes \omega_{\mathcal{A}})U^{\dagger}(r\epsilon)]$, where now $U(r\epsilon) = \mathcal{T} e^{-i \int_0^{r\epsilon} H(s)ds}$. We then consider $\frac{1}{\epsilon}\{\rho[(r + 1)\epsilon] - \rho(r\epsilon)\}$ and expand $U(r\epsilon)$ to second order (cf. [27]). In our case this is necessary because the field operators contain a factor of $\sqrt{\epsilon}$. In the continuum limit where $n \to \infty$ and $\varepsilon \to 0$, we arrive at a differential equation for $\rho(x)$ with $x \in [0, L]$,

$$\frac{d\rho}{dx} = -i[K, \rho] - \frac{1}{2}\sum_{j=1}^{4}([M_j^{\dagger}M_j, \rho]_+ - 2M_j\rho M_j^{\dagger}),$$

where in this expression all operators are evaluated at position x, i.e., $K = K(x)$, $\Psi = \Psi(x)$, etc., This is an example of a generator of dissipative dynamics which are manifestly completely positive. Here, the Lindblad operators are identified as $M_1 = iaR - bR^{\dagger}$, $M_2 = iaR + bR^{\dagger}$, $M_3 = cR^{\dagger}$, and $M_4 = dR$, where $a^2 = \langle(\Psi^{\dagger})^2\rangle/2$, $b^2 = \langle(\Psi)^2\rangle/2$, $c^2 = \langle\Psi^{\dagger}\Psi\rangle$, and $d^2 = \langle\Psi\Psi^{\dagger}\rangle$. We write this equation as $\rho'(x) = \mathcal{L}_x[\rho(x)]$.

We now describe a key feature of cMPS, shared by MPS [28], namely, their holographic property: expectation

values of field operators may be obtained in terms of the dissipative dynamics of the boundary zero-dimensional field $\mathcal{B}$ alone. This is strongly reminiscent of the holographic principle [29]. This is easy to establish using the following calculational principles (cf. [16]). Let A be any observable on $\mathcal{A}$ which is some product of the field operators and their derivatives at locations $x_1, \ldots, x_n \in [0, L]$. The first step is to put the observable into normal order with all field annihilation operators on the right. Now we must calculate $\langle A \rangle = \mathrm{tr}\{(A \otimes \mathbb{I}_{\mathcal{B}})U(L) \times [\rho(0) \otimes \omega_{\mathcal{A}}]U^{\dagger}(L)\}$. To eliminate the field operators we exploit the formula $[\Psi(x), U(L)] = -i\int_0^L U(L-s) \times [\Psi(x), F(s)]U(s)ds$, where $F(s) = Q(s) + R(s) \otimes \Psi^{\dagger}(s)$, to commute all field annihilation operators past $U(L)$. In the case that (similar results apply in the finite filling case) $\Psi(x)|\Omega\rangle_{\mathcal{A}} = 0$ we thus learn that $\Psi(x)U(L)|\Omega\rangle_{\mathcal{A}} = U(L-x)R(x) \times U(x)|\Omega\rangle_{\mathcal{A}}$. Similarly, to evaluate derivatives of Ψ we follow the same procedure with an additional integration by parts to eliminate the derivative of the delta function. We find $\Psi'(x)U(L)|\Omega\rangle_{\mathcal{A}} = U(L-x) \times \{-[Q(x), R(x)] + R'(x)\}U(x)|\Omega\rangle_{\mathcal{A}}$. To proceed, it is expedient to employ the representation where operators $M = \sum_{j,k} m_{j,k}|j\rangle\langle k|$ are identified with quantum state vectors via $|M\rangle = \sum_{j,k} m_{j,k}|j, k\rangle$. In this way, e.g., Eq. (3), with $\langle \Psi^{\dagger 2}\rangle = \langle \Psi^2 \rangle = \langle \Psi^{\dagger} \Psi \rangle = 0$, and $\langle \Psi \Psi^{\dagger} \rangle = 1$, is written as $\frac{d}{dx}|\rho(x)\rangle = L(x)|\rho(x)\rangle$, where $L(x) = -iK \otimes \mathbb{I} + i\mathbb{I} \otimes K^T - \frac{1}{2}(R^{\dagger}R \otimes \mathbb{I} - 2R \otimes \bar{R} + \mathbb{I} \otimes R^T \bar{R})$. Thus, to evaluate a correlation function $\mathrm{tr}[A\sigma_{\mathcal{A}}]$ we simply have to integrate Eq. (3) with additional insertions of the operators $R(x) \otimes \mathbb{I}$ at the locations of Ψ, $\mathbb{I} \otimes \bar{R}(x)$ at the locations of $\Psi^{\dagger}$, $\{-[Q(x), R(x)] + R'(x)\} \otimes \mathbb{I}$ at the locations of Ψ', etc. We have thus completely eliminated the field $\mathcal{A}$.

To simulate the dynamics of a system within the cMPS class we need to accommodate the time dimension. This is straightforwardly achieved by allowing K and R to depend on t. This gives rise to a one-parameter family of a cMPS $\sigma_{\mathcal{A}}(t)$ parametrized by $K(t)$ and $R(t)$. However, in order to solve the Schrödinger equation it is necessary to calculate the derivative with respect to time. This is easily achieved using $\frac{d}{dt}U(L, t) = \int_0^L U(L-s)[\frac{\partial}{\partial t}F(s, t)]U(s)ds$, which implies that $\mathrm{tr}[A\sigma'_{\mathcal{A}}(t)]$ can also be evaluated in terms (of an integral) of the solution of the Lindblad equation with an insertion of $[\frac{\partial}{\partial t}F(s, t)] \otimes \mathbb{I} + \mathbb{I} \otimes [\frac{\partial}{\partial t}F^T(s, t)]$ at $s = t$.

So far our states have been restricted to many particle quantum mechanics in a box of length L. To connect our discussion with (nonrelativistic) quantum field theory we have to show how to define quantum states of quantum fields on the real line $\mathbb{R}$. This turns out to be straightforward: all local observable quantities acquire a well-defined limit as $L \to \infty$; all that changes is that $\rho(x)$ is replaced by the fixed point ρ_{∞} of the Liouvillian.

For finite D, and in the translation invariant setting where R and K do not depend on x, generically, all correlation functions decay exponentially. Let us assume that the generator L has a unique zero eigenvalue and the real part of any other eigenvalue is bounded from above by $-\Delta$, $\Delta > 0$

being a gap. Since for $\Psi(x)|\Omega\rangle_{\mathcal{A}} = 0$ we have that $\langle \Psi^{\dagger}(x_1)\Psi(x_2)\rangle = \mathrm{tr}_{\mathcal{A}}[U(L-x_2)RU(x_2)(\omega_{\mathcal{A}} \otimes \rho_{\mathcal{B}})U^{\dagger}(x_1)R^{\dagger}U^{\dagger}(L-x_1)]$, we can apply the above rule for integrating the master equation, using techniques to compute two-point correlation functions for dynamical semigroups [27], to see that there exists a suitable $c > 0$ with $|\langle \Psi^{\dagger}(x_1)\Psi(x_2)\rangle| \leq ce^{-\Delta(x_2 - x_1)}$. Similarly, all other spatial correlators of our original field $\mathcal{A}$ are clustering.

The perspective offered here from the viewpoint of continuous measurement and the holographic principle allows us to easily generalize the cMPS ansatz class to $(2 + 1)$-dimensional fields (such a generalization was anticipated in [16]). Suppose we have a two-dimensional (spatial) bosonic field $\mathcal{A}$ with field operator $\Psi(x, y)$, $x, y \in [0, L]$. We introduce an auxiliary $(1 + 1)$-dimensional field $\mathcal{B}$ described by a tuple of field operators $\Phi_{\alpha}(y)$, $\alpha = 1, 2, \ldots, D$, which we think of as living vertically "at the boundary" of $\mathcal{A}$. To prepare a quantum state for $\mathcal{A}$ we work entirely analogously as to before: we initialize $\mathcal{A}$ in some fiducial state, say the vacuum $|\Omega\rangle$. We then interact an "infinitesimally thin" vertical strip of $\mathcal{A}$ and $\mathcal{B}$ according to some spatially local interaction $\mathcal{R}(0)$, where

$$\mathcal{R}(x) = i\int_0^L dy R_x(\Phi_{\alpha}(y), \Phi'_{\alpha}(y)) \otimes \Psi^{\dagger}(x, y) + \mathrm{H.c.}, \tag{3}$$

and where $R_x(\Phi_{\alpha}(y), \Phi'_{\alpha}(y))$, which may depend on the position x, is some polynomial in the field operators $\Phi_{\alpha}(y)$ and their derivatives. We then proceed by interacting infinitesimal vertical strips at locations x of $\mathcal{A}$ and $\mathcal{B}$ sequentially at times $t = x$. Interleaved between each interaction between the strip at x and $\mathcal{B}$ is an evolution of $\mathcal{B}$ according to

$$\mathcal{K}(x) = \int_0^L dy K_x(\Phi_{\alpha}(y), \Phi'_{\alpha}(y)), \tag{4}$$

where $K_x(\Phi_{\alpha}(y), \Phi'_{\alpha}(y))$ is some local Hermitian operator written in terms of the field operators $\Phi_{\alpha}(y)$ and their derivatives. This unitary process, illustrated in Fig. 1, is described by the propagator

$$U(L, L) = \mathcal{T}e^{-i\int_0^L dx \mathcal{K}(x) + [i\mathcal{R}(x) + \mathrm{H.c.}]}. \tag{5}$$

The previous analysis establishes the holographic property of the states generated by $U(L, L)$. A central role is again played by the Lindblad equation

$$\frac{d\rho}{dx} = -i[\mathcal{K}, \rho] - \frac{1}{2}\int_0^L dy[R^{\dagger}(y), R(y)\rho] + \mathrm{H.c.}.$$

This describes a local dissipative field theory for $\mathcal{B}$; expectation values of physical operators may be recovered by integrating this equation with the appropriate insertions.

In this work we discussed a physical interpretation of a recently introduced variational class, continuous matrix product states, for quantum fields. We have explained how this class arises naturally from the procedure of continuous measurement, and used this observation to explain the key
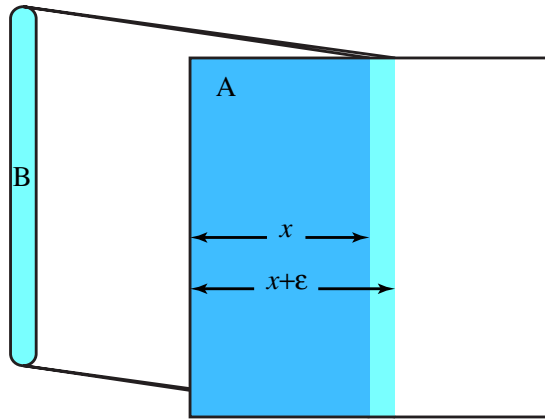
FIG. 1 (color online). Here we illustrate the physical process underlying the construction of a $(2 + 1)$-dimensional generalization. The system $\mathcal{A}$ is initialized in the vacuum state $|\Omega\rangle$ and then infinitesimally thin vertical strips at horizontal location x are sequentially interacted with $\mathcal{B}$ at time $t = x$.

physical properties of cMPS, including the clustering of correlations. We also discovered a fundamental holographic property possessed by cMPS: they can be understood in terms of a boundary field theory of one dimension lower. It is worth noting that our boundary field theory does not arise from a simple restriction of the bulk fields to the boundary, rather, it emerges in a more sophisticated way: the quantum correlations of any contiguous region in the bulk with the rest of the system will be concentrated at the boundary of the region; the boundary theory arises as the theory which parametrizes the system external to the boundary and which captures these correlations in the most parsimonious fashion possible. We have also pointed out that the zero-dimensional boundary field corresponds to the internal degrees of freedom of an atom in cavity QED experiments; this opens up the possibility of simulating quantum field theories with simple dissipative dynamics.

We believe that the interpretation of cMPS discussed here offers several advantages, particularly with respect to higher-dimensional generalizations: the manifest unitary nature of our process allows one to straightforwardly cope with superselection rules and other physical constraints and our higher-dimensional generalization can be used to apply the variational principle to, e.g., strongly interacting field theories beyond the reach of perturbation theory.

A great many future directions present themselves at this point: one can explore the utility of cMPS as a variational ansatz for numerical calculations. Finally, the relationship of the regulator that cMPS provides with other standard renormalization prescriptions remains to be elucidated.

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