Monte Carlo study of the triangular Blume-Capel model under bond randomness

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The effects of bond randomness on the universality aspects of a two-dimensional ($d = 2$) Blume-Capel model embedded in the triangular lattice are discussed. The system is studied numerically in both its first- and second-order phase-transition regimes by a comprehensive finite-size scaling analysis for a particularly suitable value of the disorder strength. We find that our data for the second-order phase transition, emerging under random bonds from the second-order regime of the pure model, are compatible with the universality class of the two-dimensional (2D) random Ising model. Furthermore, we find evidence that, the second-order transition emerging under bond randomness from the first-order regime of the pure model, belongs again to the same universality class. Although the first finding reinforces the scenario of strong universality in the 2D Ising model with quenched disorder, the second is in difference from the critical behavior, emerging under randomness, in the cases of the ex-first-order transitions of the Potts model. Finally, our results verify previous renormalization-group calculations on the Blume-Capel model with disorder in the crystal-field coupling.

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I. INTRODUCTION

The effect of quenched randomness on the equilibrium and dynamic properties of macroscopic systems is a subject of great theoretical and practical interest. It is well known that quenched bond randomness may produce drastic changes on phase transitions depending on the type of the transition [1–8]. Thus, symmetry-breaking first-order transitions are converted to second-order phase transitions by infinitesimal bond randomness for spatial dimensionality $d = 2$ [3,4] and by bond randomness beyond a threshold strength in $d > 2$ [4], as indicated by general arguments [5], rigorous mathematical work [3], and physically intuitive mappings to other disordered systems [7]. In particular, this rounding effect of first-order transitions has now been rigorously established in a unified way in low dimensions ($d \leqslant 2$) including a large variety of types of randomness in classical and quantum spin systems [9].

Historically, the effects of disorder on phase transitions have been studied in two extreme cases, i.e., in the limits of weak and strong (near the percolation point) disorder. The first important conjecture, known today as the Harris criterion [1], relates the value of the specific heat exponent α in a continuous transition with the expected effects of uncorrelated weak disorder in ferromagnets. According to the Harris criterion, for continuous phase transitions with a negative exponent α, the introduction of weak randomness is expected to be an irrelevant field and the disordered system to remain in the same universality class. On the other hand, the weakly disordered system is expected to be in a different universality class in the case of a pure system having a positive exponent α. Pure systems with a zero specific heat exponent ($\alpha = 0$) are marginal cases of the Harris criterion and their study, upon the introduction of disorder, has been of particular interest. The paradigmatic model of the marginal case is, of course, the general random 2D Ising model and this model has been extensively debated throughout the years [10–31]. Several recent studies, both analytical (renormalization-group and conformal field theories) and numerical [mainly Monte Carlo (MC) simulations] devoted to this model, have provided very strong evidence in favor of the so-called scenario of logarithmic corrections. According to this, the effect of infinitesimal disorder gives rise to a marginal irrelevance of randomness and besides logarithmic corrections, the critical exponents maintain their 2D Ising values [13–16].

An interesting candidate for investigating the above predictions in $d = 2$—apart from the well-studied case of the q-states Potts model [20,32–36]—is the Blume-Capel (BC) model [37,38]. Historically, the pure version of the BC model, consisting of a spin-one Ising Hamiltonian with a single-ion uniaxial crystal-field anisotropy [see Eq. (1) below], has been served as one of the most studied models in the communities of statistical mechanics and condensed matter physics. This is not only because of the relative simplicity with which approximate calculations for this model can be carried out and tested, as well as the fundamental theoretical interest arising from the richness of its phase diagram, but also because versions and extensions of the model can be applied for the description of many different physical structures, i.e., multicomponent fluids, ternary alloys, and ^{3}He-^{4}He mixtures [39]. It is worth noting, the latest applications of the BC model include analyses of ferrimagnets, as recently discussed by Selke and Oitmaa [40]. In the present study we are interested in investigating the effects of disorder on the critical behavior and relevant universality aspects of the BC model. In this framework, the

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advantage of dealing with the BC model stems from the fact that its phase diagram consists, in the pure version, of continuous Ising-like transitions to an ordered ferromagnetic phase as the temperature is lowered for crystal-field couplings less than a tricritical value and a first-order transition for larger values of the crystal-field coupling. Therefore, the BC model clearly offers the opportunity to study, on the same footing, two important aspects on the effects of disorder on critical phenomena, namely, the marginal case of the Harris criterion in the regime where the 2D BC model is in the same universality class with the Ising model and also the softening of the transition in the corresponding first-order regime.

The pure BC model [37,38] is defined by the Hamiltonian

$$\mathcal{H}_p = -J \sum_{\langle ij \rangle} s_i s_j + \Delta \sum_i s_i^2, \tag{1}$$

where the spin variables s_i take on the values -1, 0, or $+1$, $\langle ij \rangle$ indicates summation over all nearest-neighbor pairs of sites, and $J > 0$ is the ferromagnetic exchange interaction. The parameter Δ is known as the crystal-field coupling and to fix the temperature scale we set $J = 1$ and $k_B = 1$. As is well known, this model has been analyzed, in addition to the original mean-field theory [37,38], by a variety of approximations and numerical approaches in $d = 2$ and $d = 3$ [41–52]. As already mentioned, the Δ-T phase diagram of the model consists of a segment of continuous Ising-like transitions at high temperatures and low values of the crystal field which ends at a tricritical point $[\Delta_t;\ T_t]$, where it is joined with a second segment of first-order transitions between $[\Delta_t\ ;\ T_t]$ and $[\Delta = zJ/2;\ T = 0]$, where z defines the coordination number of the lattice. The 2D model given by Eq. (1) is studied here on the triangular lattice ($z = 6$) and will be referred to as the pure BC model. For this case, the tricritical value of the crystal field has been estimated to be $\Delta_t = 2.925(8)$ [51]. In Fig. 1, an approximation of the phase diagram of the model is given, as was estimated recently by extensive MC simulations for several values of the crystal-field coupling Δ [51].

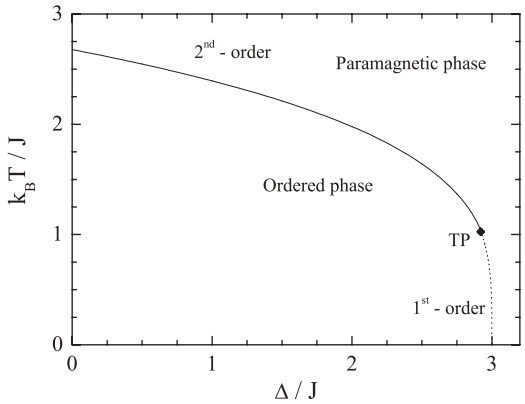

FIG. 1. Approximation of the phase diagram of the pure triangular BC model. The dotted and solid curves correspond, respectively, to first- and second-order phase-transition lines that separate the ordered and paramagnetic phases. The black rhombus points out the approximate location of the tricritical point (TP).

However, our main focus is the case with bond disorder given by the bimodal distribution

$$\mathcal{P}(J_{ij}) = \frac{1}{2}[\delta(J_{ij} - J_1) + \delta(J_{ij} - J_2)];$$
$$\frac{J_1 + J_2}{2} = 1; \quad J_1 > J_2 > 0; \quad r = \frac{J_2}{J_1}, \tag{2}$$

so that r reflects the strength of the bond randomness. In Eq. (2) we keep the ratio of interaction strengths in a fixed 50%:50% weak:strong bond mixing, following the usual practice of the traditional bond disorder implementation in the case of the square lattice Ising and Potts models [7,10]. Although in these models the above choice is a clear asset for the analysis, as the critical temperature of the system as a function of the disorder $[T_c = T_c(r)]$ is exactly known through duality relations, in the present BC case the only practical advantage is an easier implementation of disorder in the lattice. The resulting quenched disordered (random-bond) version of the Hamiltonian defined in Eq. (1) reads now as

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} s_i s_j + \Delta \sum_i s_i^2. \tag{3}$$

The rest of the paper is laid out as follows: In the following section we give a description of our numerical approach utilized to derive numerical data for large ensembles of disorder realizations and triangular lattices with up to $N = 240 \times 240$ spins. In Sec. III the finite-size scaling (FSS) analysis of the numerical data is presented, discussing the effects of bond disorder in both the originally second- (Sec. III A) and first-order (Sec. III B) phase-transition regimes of the model. Finally, in Sec. IV we summarize our conclusions and critically discuss the results of our contribution under the prism of the current literature.

II. SIMULATION PROTOCOL AND PHYSICAL REMARKS

As already well established in the relevant literature, numerical simulations are crucial to achieve progress in the field, especially when it comes to disordered systems. It is also well known that for such complex systems traditional methods become inefficient and thus in the last few years several sophisticated algorithms, some of them based on entropic iterative schemes, have been proven to be very effective [53]. The present numerical study has been carried out by applying an efficient entropic sampling scheme in two stages [10], based on the Wang-Landau (WL) algorithm [54]. One basic ingredient of this implementation is a suitable restriction of the energy subspace for the implementation of the WL algorithm. This was originally termed as the critical minimum energy subspace restriction [55] and it can be carried out in many alternative ways, the simplest being that of observing the finite-size behavior of the tails of the energy probability density function of the system [55].

Complications that may arise in complex systems, i.e., random systems or systems showing a first-order phase transition, can be easily accounted for by various simple modifications that take into account possible oscillations in the energy probability density function and expected sample-to-sample fluctuations of individual realizations [57]. Details of various sophisticated routes for the identification of the

appropriate energy subspace (E_1, E_2) for the entropic sampling of each realization have been presented in Ref. [10]. To estimate the appropriate subspace from a chosen pseudocritical temperature one should be careful to account for the shift behavior of other important pseudocritical temperatures and extend the subspace appropriately from both low- and high-energy sides, in order to achieve an accurate estimation of all finite-size anomalies. Of course, taking the union of the corresponding subspaces ensures accuracy for the temperature region of all studied pseudocritical temperatures.

The up to date version of our implementation uses a combination of several stages of the WL process. First, we carry out a starting (or preliminary) multirange (multi-R) stage, in a very wide energy subspace. This preliminary stage is performed up to a certain level of the WL random walk. The WL refinement is $G(E) \rightarrow f G(E)$, where $G(E)$ is the density of states (DOS) and we follow the usual modification factor adjustment $f_{j+1} = \sqrt{f_j}$ and $f_1 = e$ [54,55]. The preliminary stage may consist of the levels: $j = 1, \ldots, j = 18$ and to improve accuracy the process may be repeated several times. However, in repeating the preliminary process and in order to be efficient, we use only the levels $j = 13, \ldots, 18$ after the first attempt, using as the starting DOS the one obtained in the first random walk at the level $j = 12$. From our experience, this practice is almost equivalent to simulating the same number of independent WL random walks. Also in our recent studies we have found out that it is much more efficient and accurate to loosen up the originally applied very strict flatness criteria [55]. Thus, a variable flatness process starting at the first levels with a very loose flatness criteria and assuming at the level $j = 18$ the original strict flatness criteria is nowadays used. After the above described preliminary multi-R stage, in the wide energy subspace, one can proceed in a safe identification of the appropriate energy subspace using one or more alternatives outlined in Ref. [55].

The process continues in two further stages (two-stage process), using now mainly high iteration levels, where the modification factor is very close to unity and there is no significant violation of the detailed balance condition during the WL process. These two stages are suitable for the accumulation of histogram data (for instance, energy-magnetization histograms), which can be used for an accurate entropic calculation of nonthermal thermodynamic parameters, such as the order parameter and its susceptibility [55]. In the first (high-level) stage, we follow again a several times repeated (typically $\sim$5–10) multi-R WL approach, carried out now only in the restricted energy subspace. The WL levels may now be chosen as $j = 18, 19, 20$ and as an appropriate starting DOS for the corresponding starting level the average DOS of the preliminary stage at the starting level may be used. Finally, the second (high-level) stage is applied in the refinement WL levels $j = j_i, \ldots, j_i + 3$ (typically $j_i = 21$), where we usually test both a one-range (one-R) or a multi-R approach with large energy intervals. In the case of the one-R approach we have found it very convenient and more accurate to follow the Belardinelli-Pereyra adjustment of the WL modification factor according to the rule $\ln f \sim t^{-1}$, where t denotes the MC time [56]. Finally, it should also be noted that by applying in our scheme a separate accumulation of histogram data in the starting multi-R stage (in the wide energy subspace) offers the opportunity to inspect the behavior of all basic thermodynamic functions in an also wide temperature range and not only in the neighborhood of the finite-size anomalies.

A last general comment on the WL method concerns the fact that the WL recursion violates the detailed balance from the early stages of the process and care is necessary in setting up a proper protocol of the recursion. In spite of the fact that the WL method has produced very accurate results in several models, it is fair to say that there is no safe way to access possible systematic deviations in the general case. This has been pointed out and critiqued in a recent review by Janke [58]. However, from our experience and especially from our recent studies on disordered spin models [10,49,57,59], the WL implementation followed in these papers has produced excellent results, enabling at the same time the discrimination between competing theoretical predictions [10] and clarifying long debated issues in the literature [57].

As our primary goal in this paper was to identify the effect of bond disorder on the two different regimes of the pure model's phase diagram (first- and second-order phase-transition regimes in Fig. 1), the following issue should be treated with caution: Two characteristic values of the crystal-field coupling Δ should be chosen and at the same time a value of the disorder strength r, under which the originally first-order phase transition of the system undoubtedly switches to second order. In the current model this is a nontrivial issue, compared to the case of the Potts model, for which Picco [60] and Chatelain and Berche [61] have proposed techniques which allow one to find an optimal value for the ratio of couplings to secure the minimum scaling corrections to the critical behavior.

In fact, as has already been discussed in previous papers (see, e.g., Ref. [49]), there is a certain critical disorder strength, below which the BC system departs from the ferromagnetic ground state and an unsaturated ground state is produced, which is further enhanced with vacant sites ($s_i = 0$), as we increase the disorder strength. In the presence of bond randomness, the competition between the ferromagnetic interactions with the crystal-field coupling results in a destabilization of the ferromagnetic ground state. Entering in this rough regime, may cause serious problems of physical (huge increase of the sample-to-sample fluctuations) or technical origin in the positive-temperature WL simulations, which are already a difficult task due to the very dense energy states of the disordered BC model. Noteworthy here is that, as the originally ex-first-order Δ regime of the triangular BC model is rather small ($\Delta > 2.92$; see Fig. 1) the option for the choice of the crystal-field coupling in this regime is rather limited, as we do not wish to approach the regime $T \rightarrow 0$ (through $\Delta \rightarrow 3$), where the WL algorithm shows unconventional behavior.

After taking all these physical and technical restrictions into account simultaneously, and via a preliminary numerical scanning of the phase diagram of the model treating both parameters Δ and r on the same footing, we decided to select the following optimum set of values for the disorder strength and crystal-field couplings: $r = 0.75/1.25 = 0.6$, $\Delta = 1$ (ex-second-order regime), and $\Delta = 2.95$ (ex-first-order regime). For this set of values, as will be seen below, we were able to account for the logarithmic corrections that are present

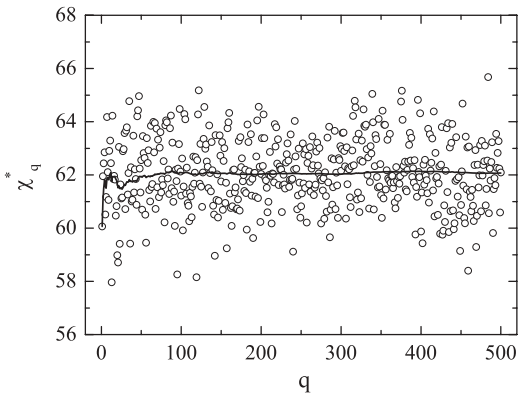

FIG. 2. Disorder distribution of the susceptibility maxima of a lattice with linear size $L = 80$ of the random-bond triangular BC model at $[\Delta = 1;\ r = 0.6]$. The running average over the samples is shown by the solid line.

in the ex-second-order regime of the model, as well as for the clear conversion of the originally first-order transition of the pure model to second order, avoiding the above discussed pathologies. Thus, using the two-stage approach we performed extensive simulations of the random BC model using lattice sizes in the range $L = 20$–240 and averaging over fairly large ensembles of random realizations $\{1, \ldots, q, \ldots, Q\}$ of the order of $Q = 250$–500. Each disorder realization was simulated approximately five times with different initial conditions to improve accuracy.

It is well known that, extensive disorder averaging is necessary when studying random systems, where usually broad distributions are expected leading to a strong violation of self-averaging [62,63]. Figure 2 presents evidence that the above number of random realizations is sufficient in order to obtain the true average behavior and not a typical one. In particular, we plot in this figure (for a lattice size $L = 80$ of the random-bond BC model at $[\Delta = 1;\ r = 0.6]$) the disorder distribution of the susceptibility maxima χ^*_q and the corresponding running average. This is a series of averages of different subsets of the full data set—each of which is the average of the corresponding subset of a larger set of data points, over the samples for the simulated ensemble of $Q = 500$ disorder realizations. A first striking observation from this figure is the existence of a very large variance of the values of χ^*_q, indicating the expected violation of self-averaging for this quantity. This figure illustrates that the simulated number of random realizations is sufficient in order to probe correctly the average behavior of the system, since already for $Q \approx 200$ the average value of χ^*_q appears quite stable.

Closely related to the above issue of self-averaging in disordered systems is the manner of averaging over the disorder. This nontrivial process may be performed in two distinct ways when identifying the finite-size anomalies, such as the peaks of the magnetic susceptibility. The first way corresponds to the average over disorder realizations ($[\ldots]_{\rm av}$) and then taking the maxima ($[\ldots]^*_{\rm av}$) and identifying their temperature locations $T_{[\ldots]^*_{\rm av}}$. However, in an alternative approach [63] one may consider individual sample dependent anomalies ($[\ldots^*]_{\rm av}$) and the corresponding sample dependent pseudocritical temperatures $[T^*_{\ldots}]_{\rm av}$, as also followed in the current investigation. This alternative route is far more demanding computationally, but the corresponding FSS analysis may be more precise, and additional useful information concerning the properties of disorder averages becomes available. It should be noted here that, for disordered systems one can make a clear distinction between typical and averaged exponents [64,65]. An extreme example of this case with very pronounced differences in the corresponding ν exponents has been provided, and critically discussed, by Fisher [64] on the random transverse-field Ising chain model. Finally, disordered critical phenomena are known to display in general multicritical exponents [66,67].

Closing this outline, let us comment on the statistical errors of our numerical data. The statistical errors of our WL scheme on the observed average behavior were found to be of small magnitude (of the order of the symbol sizes) and thus are neglected in the figures. On the other hand, the error bars shown in the figures below reflect the sample-to-sample fluctuations of the disorder-averaged maxima $[\ldots^*]_{\rm av}$.

III. FINITE-SIZE SCALING ANALYSIS

A. Ex-second-order phase-transition regime

We now present our numerical results and the relevant finite-size FSS analysis for the random-bond 2D triangular BC model with $\Delta = 1$ for disorder strength $r = 0.6$. Bond randomness favoring second-order transitions, this system is also expected to undergo a second-order transition between the ferromagnetic and paramagnetic phases and it is reasonable to expect that this transition will be in the same universality class as the 2D random Ising model. The latter model is a particular case of the more general random Ising model (random site, random bond, and bond diluted) and has been extensively investigated and debated [10–30]. As already discussed in the Introduction, using renormalization-group and conformal field theories, the marginal irrelevance of randomness at the second-order ferromagnetic-paramagnetic transition has been predicted for the marginal case of the 2D random Ising model [13–16]. More recently, this appearance of logarithmic corrections in the scaling behavior of the thermodynamic quantities that characterize the transition and its implications in the scaling analysis have been discussed under a new prism by Berche and Shchur [68], as well as Kenna *et al.* [69]. In the following we will also consider these logarithmic corrections in the relevant fitting attempts of our numerical data for the ex-second-order regime of the random BC model.

Figure 3 illustrates in the main panel the shift behavior of six disorder-averaged pseudocritical temperatures as a function of the inverse linear size $1/L$. The pseudocritical temperatures considered correspond to the peaks of the following six quantities: specific heat C, magnetic susceptibility χ, derivative of the absolute order parameter with respect to inverse temperature $K = 1/T$: $\partial\langle|M|\rangle/\partial K = \langle|M|\mathcal{H}\rangle - \langle|M|\rangle\langle\mathcal{H}\rangle$ [70], and logarithmic derivatives of the first- ($n = 1$), second- ($n = 2$), and fourth-order ($n = 4$) powers of the order parameter with respect to inverse temperature $\partial \ln\langle M^n\rangle/\partial K = \langle M^n\mathcal{H}\rangle/\langle M^n\rangle - \langle\mathcal{H}\rangle$ [70].

Fitting simultaneously our data for the lattice range $L = 40$–240 to the expected power-law behavior, including the

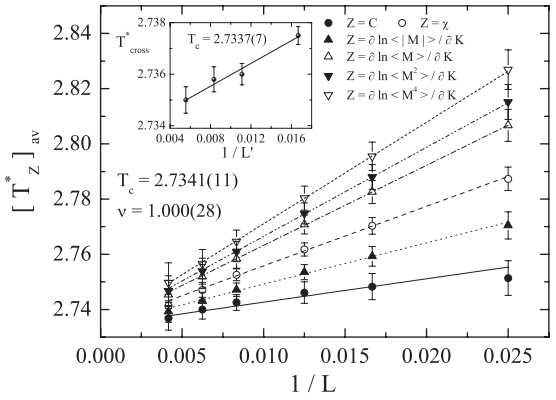

FIG. 3. Estimation of the critical temperature and correlation length's exponent of the random-bond triangular BC model at [$\Delta = 1$; $r = 0.6$]. Shift behavior of several pseudocritical temperatures defined in the text (main panel). Additional estimation of the critical temperature via the crossings of the fourth-order Binder's cumulant (inset).

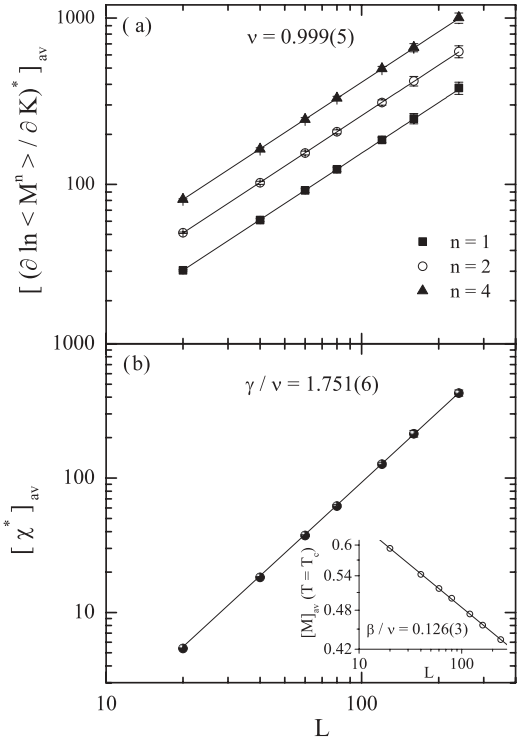

FIG. 4. Estimation of critical exponents ν [upper panel (a)], γ/ν [main lower panel (b)], and β/ν [inset of lower panel (b)] of the random-bond triangular BC model in the ex-second-order regime [$\Delta = 1$; $r = 0.6$]. The data are shown in a double logarithmic scale.

theoretically predicted logarithmic corrections $[T_Z^*]_{\rm av} = T_c + b_Z L^{-1/\nu + c_z [\ln(L)]^{-\omega}}$ [13], where Z stands for the different thermodynamic quantities mentioned above, we find the critical temperature of the random model to be $T_c = 2.7341(11)$ and the estimate $\nu = 1.000(28)$ for the critical exponent of the correlation length. Note that in the above fitting attempt we have fixed the correction-exponent ω to the value $\omega = 1$, as also indicated by Talapov and Shchur in their relevant scaling analysis of the random Ising model [18]. In fact, by performing a set of different fittings, we identified that this value for ω gave the best-fitting quality and estimates for the expected value $\nu = 1$. The inset of Fig. 3 is a further illustration of the accuracy of our numerical scheme. We plot the FSS of the crossings of the fourth-order's Binder cumulant with the inverse new linear size L' (see definition below). We show four data points which denote the temperature crossing points ($T^*_{\rm cross}$) of the Binder cumulant for the following pairs of lattices: $(L_1, L_2) =$ (40,80), (60,120), (80,160), and (120,240). The notation L' in the x axis refers to the value $L' = (L_1 + L_2)/2$. The solid line is a linear fitting extrapolating to $L' \to \infty$, which gives an estimate for the critical temperature $T_c = 2.7337(7)$ in excellent agreement with the estimate shown in the main panel of the same figure.

In Fig. 4 we present an alternative estimation of the critical exponent ν [upper panel (a)] and original estimations of the magnetic exponent ratios γ/ν [main lower panel (b)] and β/ν [inset of lower panel (b)] for the case $\Delta = 1$ of the random ($r = 0.6$) BC model. In particular, in panel (a) we illustrate the FSS of the maxima of the disorder-averaged order-parameter's logarithmic derivatives of first ($n = 1$), second ($n = 2$), and fourth order ($n = 4$), specifically defined above. These quantities are expected to scale, in a second-order phase transition, as $\sim L^{1/\nu}$ with the system size [33,70]. Including here the logarithmic corrections, we get $[(\partial \ln\langle M^n\rangle/\partial K)^*]_{\rm av} = a_n L^{1/\nu + b_n [\ln(L)]^{-\omega}}$, following the practice of the shift scaling form of the pseudocritical temperatures. n refers as usual to the order of the order-parameter derivative. A simultaneous fitting for all numerical data gives the estimate $\nu = 0.999(5)$ in very good agreement with the expected value $\nu = 1$ and with the original estimation via the shift behavior of Fig. 3.

Moving now to the main bottom panel (b) of Fig. 4, the FSS behavior of the maxima of the disorder-averaged magnetic susceptibility for the complete lattice range is shown. This is expected to scale as $[\chi^*]_{\rm av} = a L^{\gamma/\nu + b[\ln(L)]^{-\omega}}$ with the system size. The fitting procedure provides us with the estimate $\gamma/\nu = 1.751(6)$, clearly identical to the value 1.75 of the pure Ising model. In the corresponding inset of panel (b) we illustrate the scaling behavior of the magnetization data at the estimated critical temperature $T_c = 2.747$. These values were extracted from the disorder-averaged curves of the order parameter as a function of the temperature obtained in our WL simulations. Note here that, for every disorder realization the curve $M = M(T)$ is a direct outcome of the WL algorithm for a wide temperature range, with no additional cost, and this is one of its clear advantages in the numerical simulations of disordered systems [57]. The applied power law $[M]_{\rm av}(T = T_c) = a L^{-\beta/\nu + b[\ln(L)]^{-\omega}}$, shown by the solid line in a double logarithmic scale, produces an estimate $\beta/\nu = 0.126(3)$, almost identical to the value 0.125 of the Ising model. Let us comment here that these values of the ratios γ/ν and β/ν are obeyed, not only in the simple Ising model, but also in several other cases in $d = 2$. In particular, it appears that they are very well obeyed in the cases of disordered models, including the site-diluted, bond-diluted, and random-bond Ising model [10,21,23]. Furthermore, it has been shown that they are also valid in both the pure and random-bond versions of the square Ising model with nearest- and next-nearest-neighbor competing interactions [10].

At this point we would like to draw attention to a relevant paper by Aarão Reis *et al*. [22]. These authors implemented extensive transfer-matrix calculations to address the notions of strong universality and scaling corrections in disordered systems, using as a toy model the 2D random-bond Ising model. In general, their analysis provided a clear manifestation of the strong universality scenario. Moreover, through a careful consideration of subtle FSS effects they have shown that, while the correlation length and the susceptibility display no signature of size-dependent logarithmic corrections, the temperature derivative of the correlation length shows an ln L dependence, where L defines the strip width. Their FSS theory also suggested that there exists a critical size above which logarithmic enhancements vanish, leaving only the typical pure system's power-law-like behavior. Although the main outcome of Ref. [22] on the strong universality hypothesis is fully aligned with the current results of the random BC model, a discrepancy related to the presence of logarithmic corrections in the scaling behavior of the temperature and susceptibility data of the present model is raised. To our understanding, this may be due to the different nature of the BC model, or furthermore to the considered L regime. It may be that the linear sizes studied here—and also in several other related papers, where analogous corrections have been considered [10,27,30]–are still small enough to allow a clear identification of such effects. In any case, it is well accepted that transfer-matrix calculations are by far more accurate than any kind of MC scheme and the analysis of Ref. [22] opens a route for the investigation of the random BC model under a new perspective.

To summarize, in the first part of our study we have investigated the critical properties of the triangular BC model under bond randomness in the second-order transition regime of the corresponding pure model ($\Delta = 1$). We have provided clear evidence that the random model shares in this regime the values of the simple Ising model's critical exponents and we have also shown the marginal irrelevance of disorder, as mirrored in the logarithmic corrections appearing in the scaling forms of the thermodynamic quantities studied. A further discussion on this issue follows below in Sec. IV, where we shall contrast the scaling behavior of the specific-heat data for both values of Δ considered in this paper. In the sequel, we proceed with the investigation of the effects of disorder on the ex-first-order regime of the triangular BC model.

B. Ex-first-order phase-transition regime

We discuss in this part of the paper the FSS analysis of the random ($r = 0.6$) triangular BC model in its ex-first-order regime for the value $\Delta = 2.95$ of the crystal-field coupling. As already noted above, the tricritical value of the pure model's crystal field—above which a clear first-order phase transition takes place—is approximated to be $\Delta_t = 2.925(8)$ [51], which indicates that the chosen value of $\Delta = 2.95$ goes well into the originally first-order regime of the pure model. Moreover, from our preliminary analysis, the value $r = 0.6$ of the bond randomness switches directly the transition to second order and this has been verified in a number of relevant thermodynamic quantities, such as the energy probability density function and the order parameter as a function of the temperature.

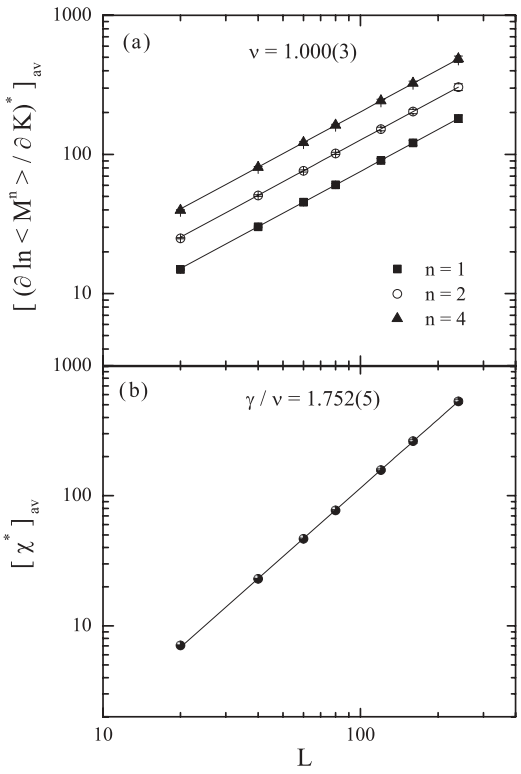

FIG. 5. The same as the main panels of Fig. 4 but for the case [$\Delta = 2.95$; $r = 0.6$] of the ex-first-order regime of the pure model.

Figure 5 illustrates the same aspects of the critical behavior of the model with that defined in Fig. 4, but for crystal-field value $\Delta = 2.95$, again in a double logarithmic scale. The results obtained from simultaneous [panel (a)] and simple [panel (b)] power-law fittings of the form $[(\partial \ln\langle M^n\rangle/\partial K)^*]_{\rm av} = a_n L^{1/\nu}$ and $[\chi^*]_{\rm av} = aL^{\gamma/\nu}$, respectively, point to an Ising-like continuous transition, with critical exponents $\nu = 1.000(3)$ and $\gamma/\nu = 1.752(5)$. Let us make a small comment at this point about the possibility of the existence of further corrections to the power laws applied in the data of Fig. 5. In principle, one would expect also for this disorder-induced second-order phase transition the presence of logarithmic corrections in the logarithmic derivatives of the order parameter and the magnetic susceptibility, as for the ones described in Sec. III A. However, performing an extensive fitting procedure to the numerical data of Fig. 5 including logarithmic corrections and several test values of the correction-exponent ω, we did not observe any kind of substantial improvement, either in the stability of the fittings or in the resulting exponent's estimate. Perhaps, in the present case of a disorder-induced continuous transition, larger systems may be necessary to clear out the role of these logarithmic corrections in quantities related to the order parameter and its fluctuations. This is an open problem that calls for further numerical tests on suitable models that undergo a first-order phase transition in their pure version, and for which vast simulations of even larger systems are easier to perform. Alternatively, the methods and analysis of Ref. [22] may be a true asset for the achievement of such a complicated task.

Comparing now the main result of Fig. 5 with the current literature on the 2D ($q > 4$) Potts model, we find a marginal

difference. Although the first work on the random $q = 8$ Potts model by Chen *et al*. [33] suggested that the emerging, under disorder, continuous transition belongs to the universality class of the Ising model, this result is known now to be false. In particular, many works have been devoted to this issue since then [7,8,36,60,61,66,67,71–77] and the values of the critical exponents are pretty well known for various values of q (below and above $q = 4$). So, the situation is now completely clarified, with the universality class of the random Potts model changing continuously with the values of q.

However, the behavior and critical properties of the current BC model, which is clearly less studied, is under debate. In terms of physical interactions the BC model is much more complicated than the corresponding Potts model, and this stems from the existence of vacant sites, $s_i = 0$, and from the presence of competition between the ferromagnetic interactions with the crystal-field coupling, a phenomenon that becomes even more pronounced under the presence of bond disorder. Thus, since the physical mechanisms that respond to the presence of randomness are quite different among the two models, we believe it is not fair to judge the models under the same prism. Of course, it would be very interesting to have a more wide spectrum of results in terms of crystal-field couplings and disorder strengths, but as already stated in Sec. II, this is a rather demanding task that goes beyond the first observations pointed out in the present paper.

IV. DISCUSSION AND OUTLOOK

Summarizing, in the present paper we have investigated through extensive simulations via an optimized numerical approach based on a two-stage implementation of the WL algorithm, the critical behavior of the $d = 2$ random-bond triangular BC model. In the first part of our analysis, referring to the originally second-order transition regime of the model, we showed that the model belongs to the universality class of the random Ising model, as expected on theoretical grounds. This has been achieved by estimating with high accuracy the set of critical exponents that describe the continuous transition of the model for $\Delta = 1$, incorporating also in the scaling analysis the proposed logarithmic corrections.

At this point we would also like to address the scaling behavior of the specific heat, which is known to be the most hard to deal with quantity within the theory of FSS, on the same footing for both values of the crystal field considered (ex-first- and ex-second-order regimes of the model). Figure 6 illustrates the scaling behavior of the disorder-averaged specific-heat maxima for $\Delta = 1$ and $\Delta = 2.95$ and for the larger lattice sizes considered ($L \geqslant 60$) as a function of the double logarithm of the lattice size. Using these data we tried to observe the quality of the fittings, assuming a logarithmic divergence, a double-logarithmic divergence, and a power-law behavior. Although there is no irrefutable way of numerically distinguishing between the above scenarios, our fitting attempts indicated that the power-law option is completely excluded, and also that the double logarithmic form applies more than satisfactorily to our data and gives stable results, as moving to higher lattice-size ranges, especially when compared to the simple logarithmic scaling behavior encountered in the 2D Ising model. Thus, in Fig. 6 we present linear fittings in both $\Delta = 1$ and $\Delta = 2.95$

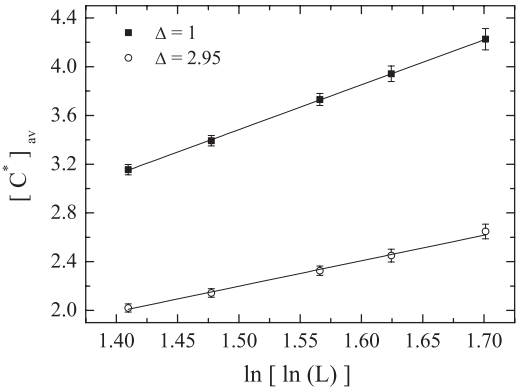

FIG. 6. Disordered averaged specific-heat data as a function of the double logarithm of the lattice size for the two values of Δ considered in this paper. The solid lines are linear fittings for $L \geqslant 60$.

data which follow the logarithmic corrections of the form $[C^*]_{\rm av} = a + b\ \ln[\ln(L)]$. This result, given that the BC model shares in its pure version—for $\Delta < \Delta_t$—the critical exponents of the Ising model, is a further verification in favor of the well-established logarithmic corrections' scenario proposed originally for the random version of the Ising model is $d = 2$.

Regarding now the effect of disorder on the line of first-order phase transitions of the pure model, we presented results for the emerging, under random bonds, continuous transition, that are compatible again with the Ising universality class and with a double logarithmic divergence of the specific heat. This outcome is in contradiction to relevant results concerning the emerging under disorder continuous transitions of the ($q > 4$) Potts model in $d = 2$. In fact, it is now well established in the current literature [7,8,36,60,61,66,67,71–77] that the universality class of the random Potts model is never that of the pure (or disordered) Ising model, but changes as a function of the number of states of the model. Thus, one may suspect that this should also be the case for the current BC model, i.e., the appearance of second-order phase transitions that belong to different universality classes, depending continuously on the value of the crystal-field coupling. Although such a scenario cannot be completely excluded, we believe that the extensive simulations and careful analysis performed in the current paper account for the declared result. Of course, further simulations for various values of $\Delta(> \Delta_t)$ and stronger disorder strengths $r(< 0.6)$ are welcomed and may clear out the situation to a larger extent. However, as already noted in Sec. II, this is not an easy task. First, numerical methods that work efficiently in the low-temperature regime should be implemented. Additionally, we remind the reader that the BC model is unique, in terms of the competing presence of ferromagnetic interactions and the crystal field, a phenomenon which is further enhanced by the presence of disorder. Previous experience [49] indicates that one should be really careful when deciding the set of parameters (Δ, r) to work with, a restriction that we have treated with real caution here. Until a clear remedy of this unexplored issue is made possible—for instance, by implementing powerful optimization methods—we feel it is prudent to postpone such an attempt.

Overall, our results in both the ex-first- and ex-second-order regimes of the model are in agreement with an earlier

real-space renormalization-group analysis of the 2D random BC model performed by Branco and Boechat [78]. These authors studied the corresponding model on the square lattice for a different kind of randomness, implemented via a site-dependent crystal-field coupling Δ_i of the form $\mathcal{P}(\Delta_i) = p\delta(\Delta_i + \Delta) + (1-p)\delta(\Delta_i - \Delta)$, where $p \in (0,1)$. Their analysis predicted that the complete phase diagram of the random model consists of a line of Ising-like continuous transitions, although they were not able to account for the existence, or not, of logarithmic corrections, something that became possible in the present paper through a convincing scaling analysis in the ex-second-order regime of the model.

Last but not least, it is worth comparing the present results with the recently studied critical aspects of the $d = 2$, square lattice, random-bond BC model [49]. Although the main results of our analysis corroborate well those of Ref. [49] for the ex-second-order regime of the models, there is a marginal difference in the respective ex-first-order regime. Specifically, in Ref. [49], for the continuous transition emerging under bond randomness in the ex-first-order regime of the model, a value of $\nu > 1$ has been estimated for the critical exponent of the correlation length and also a strong saturation of the disorder-averaged specific maxima has been recorded. This observed variation can only be attributed to the different topology of the considered lattice and its sensitivity to the combination of disorder and the existence of an originally first-order transition, an aspect that may not have been previously considered under this prism in the study of disordered systems, that definitely deserves more attention.

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